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An Adaptive Approach to the Control and Synchronization of Continuous-Time Chaotic Systems

Mario di Bernardo ¹

Abstract

This paper is concerned with synchronization and control of chaotic nonlinear dynamical systems. First, a unified frame for both the synchronization and the control problem is described. Then by modifying a feedback plus feedforward controller, a discontinuous strategy is synthesized which exploits the boundedness of chaotic attractors and limit cycles. Finally, an adaptive approach is investigated and an adaptive estimation law is implemented. The result, which is both global and not reliant on complete knowledge of the systems involved, is rigorously proved by means of Lyapunov Theory. An application to the synchronization of two chaotic systems is presented.

1 Introduction

In recent years the study of chaotic nonlinear dynamical systems has rapidly expanded. Initially, chaos was considered to be mostly of academic interest. However, recently applications have arisen in the most diverse disciplines [Kim 1992]. It has been pointed out that in many of these areas it is important not only to understand and describe chaotic motions but even to regulate or *control* the dynamics into some desirable motion. For example, in communications it has been suggested that chaotic systems can be used in secure transmissions [Pecora & Carroll 1991]. Chaotic lasers have been controlled, as have the chaotic diode resonator and Chua electronic circuits [Chen & Dong 1993, Itoh, Murakami & Chua 1994]. Hence controlling a chaotic system has become a very important goal and is the subject of much on-going research.

Since the early attempts at controlling chaos, much has changed and the attitude towards chaos itself has been greatly modified. At the beginning, the major research effort was spent on eliminating chaotic behaviour from nonlinear systems. Nowadays it has been pointed out that, under certain conditions, chaotic behaviour may be useful [Shinbrot, Grebogi, Ott & Yorke 1993].

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In applications, we can characterize two main problems that arise when dealing with chaotic systems: synchronization and control.

The *synchronization problem* consists of making two or more chaotic systems oscillate in a synchronized way. This seems impossible, if we think to one of the main features of chaotic behaviour, namely the sensitive dependence on initial conditions. It is, indeed, impossible either to reproduce exactly the same starting conditions or to match exactly the parameters of the two systems. Thus, any even infinitesimal change of any parameter will eventually result in divergence of orbits starting nearby each other. Nevertheless, many results have been produced that show how synchronization can be achieved under certain conditions [Pecora & Carroll 1991].

The *control problem* is generally considered as that of stabilizing the system to a periodic orbit or equilibrium point.

The two problems can be unified as shown in Section 2 of this paper; the synchronization being seen as a type of control problem in which the goal is to track the desired chaotic trajectory [Kocarev, Shang & Chua 1993].

At present several different approaches [Chen & Dong 1993], including some conventional control techniques [Abed, Wang & Chen 1994, Chen & Dong 1992, Fowler 1989, Genesio, Tesi, Wang & Abed 1993] have been successfully applied to both the problems.

In what follows, a reference model approach, which was found to be useful in [Qammar & Mossayebi 1994], is considered in order to synthesize the control or synchronization strategy. In Section 3, a feedback plus feed-forward controller is considered, which assumes perfect knowledge of the systems involved. This assumption is then removed in the following section, exploiting the fact that the reference model is chaotic. A discontinuous action is therefore substituted to the original feed-forward term to achieve the control. Then, in Section 5, an adaptive controller is designed under the main assumption that the solutions of a chaotic system, evolving either on a strange attractor or on a periodic orbit (equilibrium point), are bounded. The adaptive strategy is used here to estimate the amplitude of the discontinuous action to be implemented by the controller. Although other adaptive schemes have been successfully applied to the control and synchronization of chaotic systems [Mossayebi, Qammar & Hartley 1991], the strategy outlined in this paper refers directly to the chaotic nature of the systems involved. Moreover, the result obtained is global and control can be achieved even by applying a direct control action only to some of the system states.

Each result is proved by means of Lyapunov Theory. This has, indeed, been shown to be a very powerful tool in this context (see Wu & Chua [1994]). Finally, numerical results are presented

that show how this method can be implemented to control a Duffing oscillator and to synchronize two identical Chua circuits.

2 The statement of the problem

Given two systems :

$$\begin{cases} \dot{x}(t) = f(x(t), t) + Bu(t), & x(0) = x^0 \in \mathcal{R}^n \\ \dot{y}(t) = g(y(t), t), & y(0) = y^0 \in \mathcal{R}^n \end{cases}, \quad (1)$$

with $f, g : \mathcal{R}^n \times \mathcal{R} \rightarrow \mathcal{R}^n$ continuous and $B \in \mathcal{R}^{n \times m}$, $x(t), y(t) \in \mathcal{R}^n$, $u(t) \in \mathcal{R}^m$, the problem is to find a feedback control $u(t) = \varphi(x(t), y(t), t)$ such that

$$\lim_{t \rightarrow \infty} |x(t) - y(t)| = 0. \quad (2)$$

This problem is equivalent to that of rendering the origin a global attractor for the error system

$$\dot{e}(t) = f(x(t), t) - g(y(t), t) + Bu(t) \quad (3)$$

where $e(t) = x(t) - y(t)$.

As proposed in Kocarev et al. [1993], both synchronization of two chaotic systems and controlling a chaotic trajectory can be described by the general problem stated above. This frame is, therefore, a very powerful tool to describe a wide class of chaotic control and synchronization problems. It is relevant at this end to point out that in (3) the controller is coupled to the system through an appropriate coupling matrix. Hence, this frame includes even the case in which only some of the system states are directly controlled, allowing large flexibility in the choice of the control action.

Let now $\Pi : \mathcal{R}^n \rightarrow \text{Im}(B)$ be the orthogonal projection operator of \mathcal{R}^n onto $\text{Im}(B)$. Thus we can decompose $f - g$ as

$$f - g = \Pi(f - g) + (I - \Pi)(f - g)$$

Assumption 1 *The projection of $f - g$ on the complementary space of $\text{Im}(B)$ is linear, that is for some linear matrix L :*

$$(I - \Pi)(f(x, t) - g(y, t)) = L(x - y), \quad L \in \mathcal{R}^{n \times n}$$

Remark. For many well-known chaotic systems such as the Lorenz system or the Chua circuit, the right-hand sides consist of at least a linear component and some nonlinearity. Hence Assumption 1 is not as strong as it might seem, being verified by a large number of systems.

Under Assumption 1 (3) becomes, assuming B to be of full rank,

$$\dot{e}(t) = Le(t) + B[h(x(t), t) - l(y(t), t) + u(t)] \quad (4)$$

with

$$\begin{aligned} h &= (B^T B)^{-1} B^T f, \\ l &= (B^T B)^{-1} B^T g. \end{aligned}$$

3 A feedback controller with feedforward

3.1 A first controller

We wish, now, to choose an appropriate function $u(t)$ in order to solve the problem stated in section 2. If we recall one of the main properties of feedback, namely, the feedback linearization, we can try to achieve the control by linearizing the systems involved via a combined feedback plus feedforward action. Under Assumption 1 we can trivially prove that

Theorem 1 *If (L, B) is stabilizable then $u(t) = \varphi(x(t), y(t), t)$ with*

$$\varphi(x, y, t) = -K(x - y) - h(x, t) + l(y, t), \quad (5)$$

where $K \in \mathcal{R}^{m \times n}$ is such that all the eigenvalues of $(L - BK)$ are in the left hand side of the complex plane, will ensure (2).

Proof. Substituting (5) into (4) we get :

$$\dot{e}(t) = (L - BK)e(t)$$

and by the way in which K is chosen, it follows that the origin is asymptotically stable and so $x(t)$ approaches $y(t)$ as $t \rightarrow \infty$. \square

Remark. The control (5) does not require any particular constraint on the initial conditions of

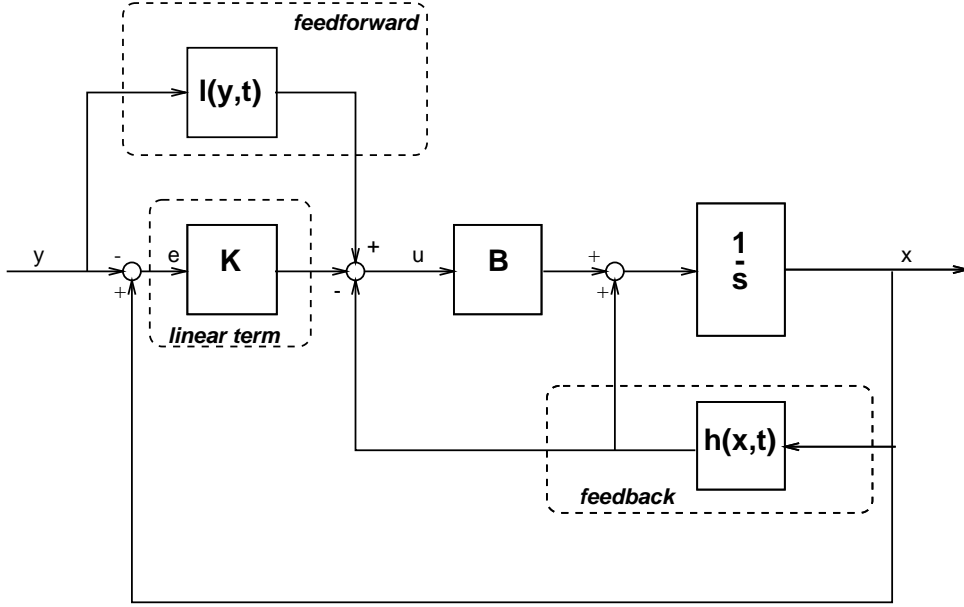


Figure 1: Ideal feedback plus feed-forward controller

systems (1). Hence the result obtained is global. The controller consists of three different contributions (see fig. 1). The error dynamics can be chosen by tuning the gain matrix K in the linear component. Two more terms, a feedback and a feed-forward, compensate the nonlinearities of the systems.

We notice that if the nonlinear compensation terms $h(x, t)$ and $l(y, t)$ were completely omitted the goal could still be achieved but an appropriate choice of initial conditions would be required, i.e. the desired trajectory would be followed provided that $\|x(0) - y(0)\|$ were sufficiently small (see Kocarev et al. [1993]).

3.2 Controlling a Duffing Oscillator.

As an example, we consider a modified Duffing Oscillator of the form :

$$\ddot{x} + p\dot{x} + p_1x + x^3 = q \cos(\omega t) + u \quad (6)$$

As shown in figure 2, with parameters $p = 0.4, p_1 = -1.1, q = 1.8$ and $\omega = 1.8$, the solution of (6) displays a chaotic response.

We now add a control term to (6) so to force the chaotic oscillator asymptotically onto the period-1 orbit (shown in fig. 3) obtained when $q = 0.620$. In order to do this, we consider a

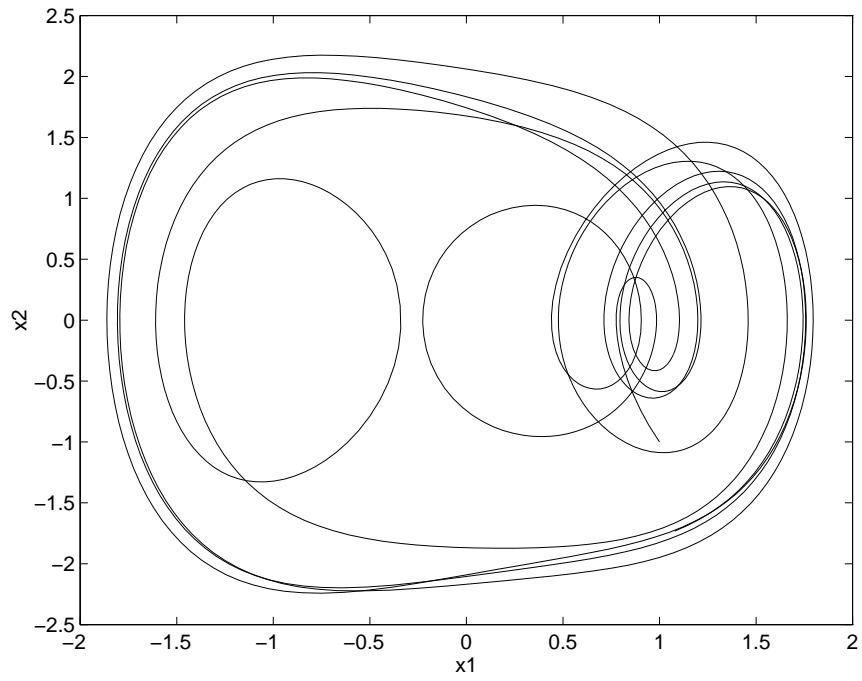


Figure 2: Chaotic orbit of the uncontrolled Duffing Oscillator

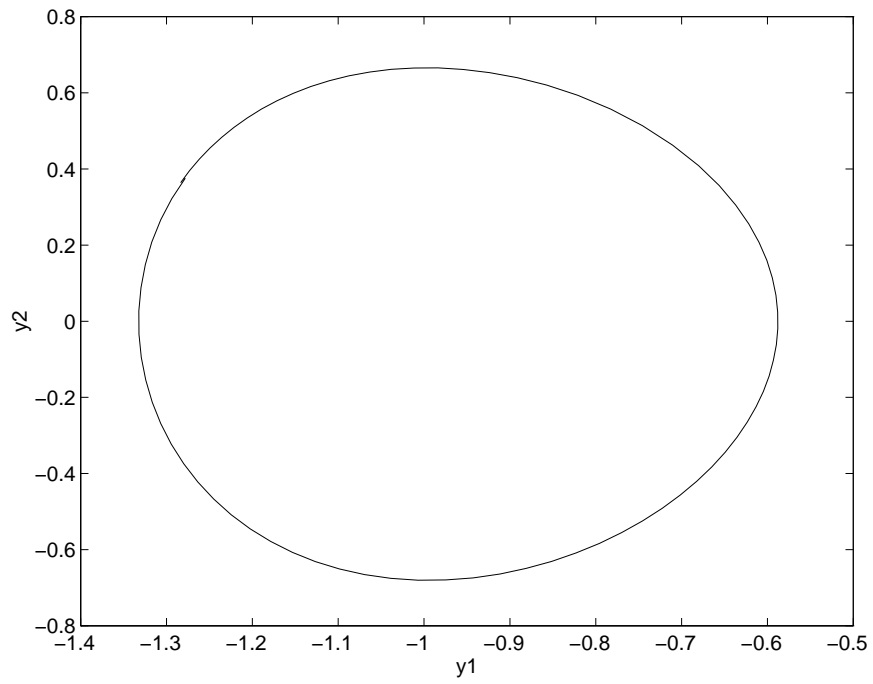


Figure 3: Periodic orbit to track ($q=0.620$)

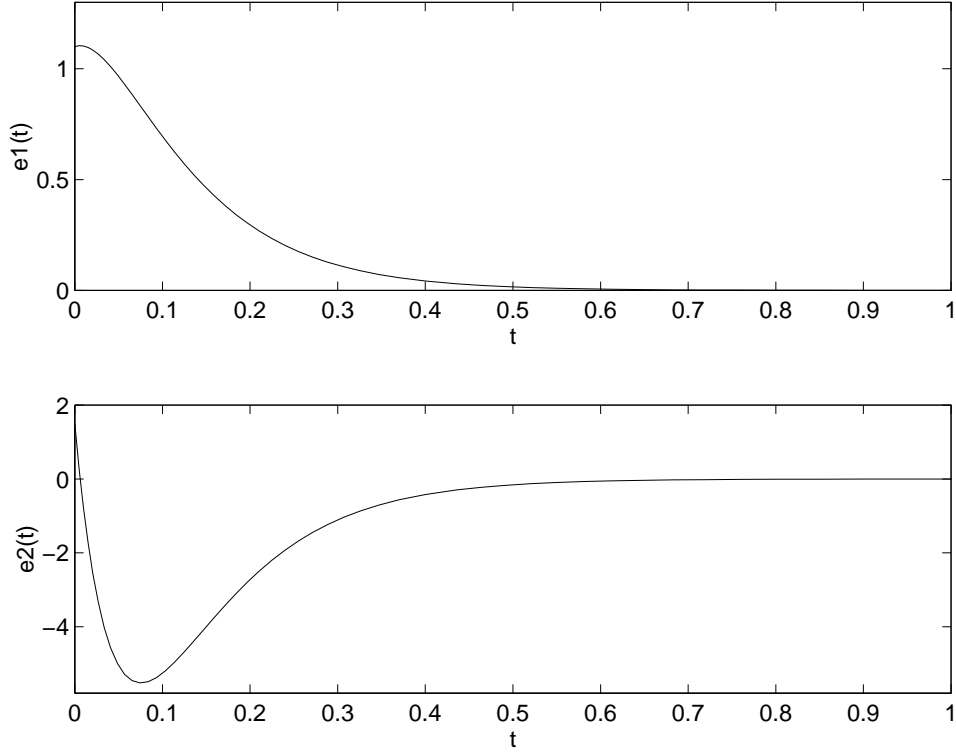


Figure 4: Controlled oscillator : error transient

second uncontrolled Duffing oscillator

$$\ddot{y} + p\dot{y} + p_1y + y^3 + q'\cos(\omega t) = 0$$

with $q' = 0.620$, which will provide the reference signal that we want to track. Rewriting each of the two systems as

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -p_1 & -p \end{pmatrix} x + \begin{pmatrix} 0 \\ -x^3 \end{pmatrix} + \begin{pmatrix} 0 \\ q \cos(\omega t) \end{pmatrix},$$

and adding the control term $\begin{pmatrix} 0 \\ 1 \end{pmatrix} u$ to the system to be controlled, we derive the error equation

$$\dot{e} = \begin{pmatrix} 0 & 1 \\ -p_1 & -p \end{pmatrix} e + \begin{pmatrix} 0 \\ y^3 - x^3 \end{pmatrix} + \begin{pmatrix} 0 \\ (q - q') \cos(\omega t) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u. \quad (7)$$

According to Theorem 1, we then choose a matrix $K = (K_{11} \ K_{12})$ such that the eigenvalues of $\hat{L} = L - BK$ are in the open left hand side of the complex plane.

Using a pole placement technique and requiring the eigenvalues of the closed loop matrix to have real parts located at -10 and -20, we get :

$$K = - (201.1 \ 29.6).$$

Hence we form the control

$$u(t) = (-201.1 \quad -29.6) e(t) + x(t)^3 - y(t)^3.$$

As shown in fig. 4, we obtain excellent dynamic behaviour. The error goes exponentially to zero, while the chaotic system approaches the reference trajectory. Finally, we add that the error dynamics can be easily changed by choosing a different gain matrix. The time scale of convergence to zero was chosen only for convenience.

4 A discontinuous controller

4.1 Exploiting chaos

The controller described in the previous section must have perfect knowledge of the systems involved and requires the availability of the full state for feedback purposes. In practical applications it is generally impossible to provide the controller with a perfect knowledge of the systems involved. Therefore, we now remove the feed-forward compensation term by analyzing the system we desire to track. We no longer suppose perfect knowledge of the function g but instead assume only that its projection l is bounded by a known continuous function γ in the following sense.

Assumption 2

$$|l(y(t), t)| \leq \gamma(y(t)), \quad \forall y, t$$

with $\gamma(y)$ continuous.

We now exploit the fact that the system $\dot{y}(t) = g(y(t), t)$ is a chaotic system evolving either on a strange attractor or on a periodic orbit/equilibrium point, to deduce that its solution must be bounded for all t . Hence, for some known $M \geq 0$, we assume

$$\|y(t)\| \leq M, \quad \forall t \in [0, \infty[, \quad M \in \mathcal{R}^+;$$

It then follows by Assumption 2 that there exists $W \in \mathcal{R}$ s.t.

$$\|l(y(t), t)\| \leq W, \quad \forall t \in [0, \infty[; \tag{8}$$

The idea is to exploit this property of the reference model, in order to achieve the control. In so doing we consider a controller of the form

$$u(t) = -K e(t) - W |B^T P e(t)|^{-1} B^T P e(t) - h(x(t), t). \tag{9}$$

where P is the positive definite matrix solution of the Lyapunov equation

$$P\hat{L} + \hat{L}^T P + I = 0.$$

Here we still have a linear term $-Ke$ and a feedback linearization term $h(x, t)$, but the feed-forward, responsible of the compensation of $g(y, t)$ has disappeared. What we have now is a discontinuous term depending upon the known bound W , which will dominate the nonlinearity of the reference model and therefore should guarantee the desired goal.

In fact, the hypothesis of Theorem 1 and (8) yield the following

Theorem 2 *Let $P \in \mathcal{R}^{n \times n}$ be the positive definite matrix solution of the Lyapunov equation*

$$P\hat{L} + \hat{L}^T P + I = 0, \quad (10)$$

with $\hat{L} = L - BK$. Define the set valued mapping $D : e \mapsto D(e) \subset \mathcal{R}^m$ such that

$$D(e) = \begin{cases} \{W\|B^T P e\|^{-1} B^T P e\}, & B^T P e \neq 0 \\ \mathcal{B}(1), & B^T P e = 0 \end{cases}, \quad (11)$$

with $\mathcal{B}(1)$ denoting a closed ball of radius one in \mathcal{R}^m , and embed the feedback controlled error system (4) in the differential inclusion

$$\dot{e} \in \{\hat{L}e + B[-l(y(t), t) + u] | u \in D(e)\} =: E(e, t). \quad (12)$$

Then the origin is asymptotically stable for the error system (4).

Proof. Note that $E(e, t)$ is upper semi-continuous on $\mathcal{R}^n \times \mathcal{R}$, with non-empty, convex and compact values. Therefore, for each $e^0 \in \mathcal{R}^n$, the system (4.5) with $e(0) = e^0$ has a maximal solution $e : [0, \omega) \mapsto \mathcal{R}^n$. Let $V(e) = \langle e, Pe \rangle$. Obviously :

$$V(e) > 0, \quad \forall e \in \mathcal{R}^n - \{0\}, \quad V(0) = 0$$

and for all $\psi \in E(e, t)$

$$\begin{aligned} \langle \nabla V(e), \psi \rangle &= -\frac{1}{2}\|e\|^2 + \langle Pe, B[-l(y) - W\|B^T P e\|^{-1} B^T P e] \rangle \\ &= -\frac{1}{2}\|e\|^2 - W\|B^T P e\|^{-1} \langle B^T P e, B^T P e \rangle + \langle Pe, B[-l(y, t)] \rangle \\ &\leq -\frac{1}{2}\|e\|^2 - W\|B^T P e\| + \|l(y, t)\| \|B^T P e\| \\ &\leq -\frac{1}{2}\|e\|^2 < 0, \quad \forall e \neq 0 \end{aligned}$$

Let $e(\cdot)$ be a maximal solution, then

$$\frac{d}{dt}V(e(t)) \leq -\frac{1}{2}\|e(t)\|^2, \quad \text{for almost all } t.$$

and we may deduce that the error decays asymptotically to zero. \square .

Remark

It is relevant to point out that in many chaotic systems the explicit dependence upon t is usually due to a bounded forcing term (frequently sinusoidal), and hence it is practically always possible to find a function $\gamma(\cdot)$ such that (8) is true.

4.2 Smoothing the discontinuous controller

As stated before, the practical disadvantage of the controller (11) is that of being discontinuous. This can be easily avoided but implies a bounded tracking error. If we consider a mapping $D' : e \mapsto D'(e)$, such that

$$D'(e) = \begin{cases} \|B^T P e\|^{-1} B^T P e, & \|B^T P e\| \geq \delta \\ \delta^{-1} B^T P e, & \|B^T P e\| < \delta \end{cases} \quad (13)$$

and develop the Proof of Theorem 2 using $D'(e)$ instead of $D(e)$, we get that, along the solutions $e(\cdot)$:

$$\frac{d}{dt}V(e) \leq \begin{cases} -\frac{1}{2}\|e\|^2, & \|B^T P e\| \geq \delta \\ -\frac{1}{2}\|e\|^2 + W\delta, & \|B^T P e\| < \delta \end{cases}.$$

Hence the error approaches the ball of radius $\|P\|\|P^{-1}\|W\delta$ enclosing the Lyapunov function level set $\|P^{-1}\|^{-1}\|e\|^2 \leq \langle e, P e \rangle \leq \|P\|\|e\|^2$. Therefore choosing an appropriate $\delta > 0$ we can make the error arbitrarily small.

4.3 Example (Duffing Oscillator)

Given the Duffing oscillator (6), we shall now try to control it using a controller of the form (11). The reference model solution is indeed pointwise bounded in norm by 2, thus we choose $W = 10$ as an upper bound for $g(y(t)) = y^3$. In order to obtain the convergence in the same time interval of the previous example, we now choose the gain matrix K to be such that the eigenvalues real parts of the closed loop matrix are located at -50 and -100. Without this gain adjustment, we will achieve the control on a longer (but still acceptable) time scale, due to the

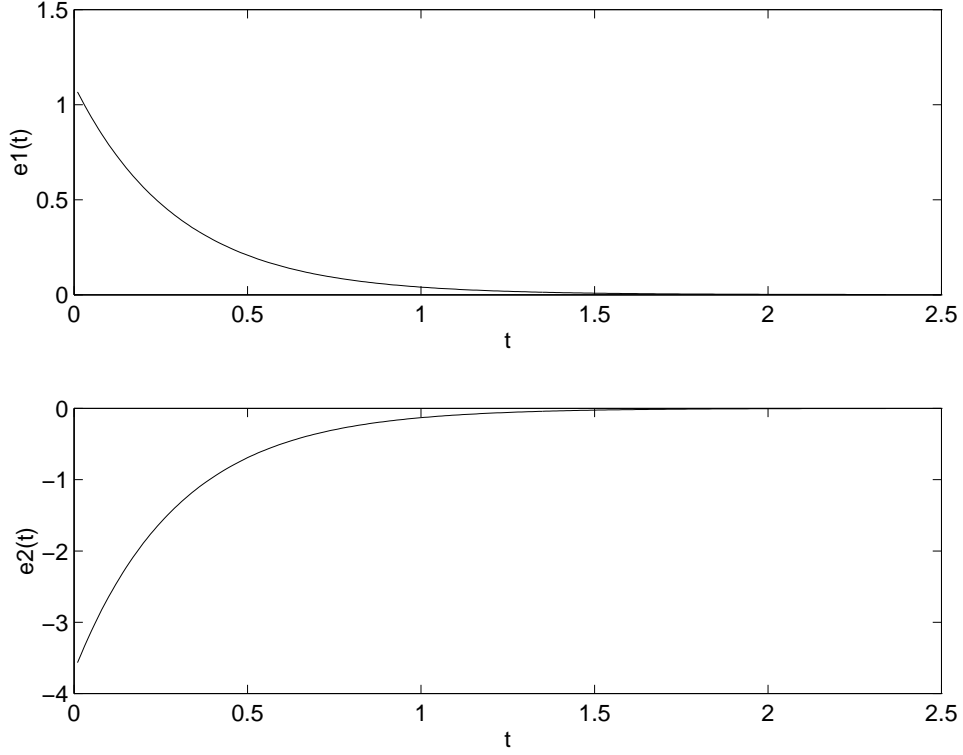


Figure 5: Error transient of the oscillator controlled according to Theorem 2

presence of the discontinuous action. Then, we compute $\hat{L} = L - BK$ and solve the Lyapunov equation (10) obtaining

$$P = \begin{pmatrix} 0.005 & 0.0051 \\ 1.068 & 0.0051 \end{pmatrix}$$

Finally, we form the control (11). Feeding the systems (1) with such a control we achieve the desired goal as shown in fig. 5.

5 An adaptive approach

5.1 W unknown

In many real situations it can be difficult to determine exactly the value of the upper bound W . Moreover when W is chosen by a worst case procedure, its value is usually a over-estimate, causing an expensive and avoidable waste of control energy.

To overcome this problem, i.e. to design a controller able to achieve the control goal, without knowing the exact value of the upper bound W , we consider an adaptive controller [Ilchmann &

Ryan 1994, Ryan 1991]. Again, the idea is that of dominating the nonlinearity of the reference model by a discontinuous term. In this case, the upper bound W is substituted by an adaptively estimated $k(t)$. Under the hypothesis of Theorem 2 we can state

Theorem 3 *Let $P \in \mathcal{R}^{n \times n}$ be the positive definite solution of*

$$P\hat{L} + \hat{L}^T P + I = 0, \quad P > 0,$$

with $\hat{L} = L - BK$.

Let

$$\dot{k}(t) = \|B^T P e(t)\|.$$

Define the mapping $D : e \mapsto D(e) \subset \mathcal{R}^m$, such that :

$$D(e) = \begin{cases} \{\|B^T P e\|^{-1} B^T P e\}, & B^T P e \neq 0 \\ \mathcal{B}(1), & B^T P e = 0 \end{cases}$$

with $\mathcal{B}(1)$ denoting a closed ball of radius one in \mathcal{R}^m and embed the feedback controlled error system (4) in the differential inclusion :

$$(\dot{e}, \dot{k}) \in \{\hat{L}e + B[-l(y(t), t) + u] | u \in kD(e)\} \times \{\|B^T P e\|\} =: E'(e, k, t). \quad (14)$$

Then, for every initial condition $(e(0), k(0)) = (e^0, k^0)$,

1. $\lim_{t \rightarrow \infty} k(t) = k^* < +\infty$;
2. $\lim_{t \rightarrow \infty} e(t) = 0$.

Proof. Note that $E'(e, k, t)$ is upper semi-continuous on $\mathcal{R}^n \times \mathcal{R} \times \mathcal{R}$ and takes non-empty, compact and convex values. Hence for each $e^0 \in \mathcal{R}^n$, $k^0 \in \mathcal{R}$, the system (14) with $e(0) = e^0$, $k(0) = k^0$ has a maximal solution $(e(\cdot), k(\cdot)) : [0, \omega[\mapsto \mathcal{R}^n \times \mathcal{R}$. Consider now the function :

$$G(e, k) = \langle e, P e \rangle + \frac{1}{2}(W - k)^2. \quad (15)$$

We have that $G(x, k)$ is greater than zero for all $(e, k) \in \mathcal{R}^n \times \mathcal{R}$.

In addition to this, differentiating (15) we get

$$\begin{aligned} \langle \nabla G(e, k), \psi \rangle &= -\frac{1}{2}\|e\|^2 + \langle P e, B[-l(y, t) - k(t)\|B^T P e\|^{-1} B^T P e] \rangle \\ &\quad - (W - k)\|B^T P e\| \\ &\leq -\frac{1}{2}\|e\|^2 - k\|B^T P e\|^{-1} \langle P e, B B^T P e \rangle + \langle P e, B[-l(y, t)] \rangle \\ &\quad - (W - k)\|B^T P e\| \\ &\leq -\frac{1}{2}\|e\|^2, \end{aligned}$$

for all $\psi \in E'(e, t)$.

Therefore, along the solution $(e(t), k(t))$

$$\dot{G}(e(t), k(t)) \leq -\frac{1}{2}\|e(t)\|^2, \quad \text{for almost all } t.$$

Hence $(e(\cdot), k(\cdot))$ is bounded and so $\omega = \infty$.

By assumption 2 and (14), it then follows that $\|\dot{e}(t)\| \leq r$, for some $r > 0$. Moreover $k(\cdot)$ is also monotone on $[0, \infty)$ and thus converges to a finite limit.

Finally let Ω denote the ω -limit set of our solution $(e(\cdot), k(\cdot))$.

Since $(e([0, \infty[), k([0, \infty[))$ is bounded it follows that Ω is not empty.

We prove the second assertion by showing that $\Omega \subset \{(e, k) \in \mathcal{R}^n \times \mathcal{R} \mid e = 0\}$.

Suppose otherwise. Then there exists $\xi = (e, k) \in \Omega$ and $\epsilon > 0$ such that $\frac{1}{2}\|\bar{e}\|^2 > 2\epsilon$. Let $\delta > 0$ be such that $\|e(t) - \bar{e}\| < \delta$ implies $\frac{1}{2}\|e(t)\|^2 > \epsilon$. By definition of an ω -limit point there exists a sequence $t_n \subset \mathcal{R}^+$ such that $(e(t_n), k(t_n)) \rightarrow (\bar{e}, \bar{k})$ as $n \rightarrow \infty$. Assume n^* to be such that $\|e(t_n) - \bar{e}\| < \frac{1}{2}\delta, \forall n > n^*$.

Then by continuity of G , for n sufficiently large,

$$G(e(t_n), k(t_n)) - G(\xi) < \frac{\epsilon\delta}{4r},$$

where $(\epsilon\delta/4r)$ is a conveniently chosen number.

Moreover on $(t_n, t_n + \frac{\delta}{3r})$ we have that $\|e(t) - \bar{e}\| < \delta$ and so $\frac{1}{2}\|e(t)\|^2 > \epsilon$. Therefore

$$G(e(t_n), k(t_n)) - G(\xi) \geq \frac{1}{2} \int_{t_n}^{t_n + \frac{\delta}{3r}} \|e(t)\|^2 dt \leq \frac{\epsilon\delta}{3r}.$$

Hence we get a contradiction. \square

5.2 Example (Duffing Oscillator)

We consider the Duffing oscillator (6) with the parameter values given there. The controller is started with $k(0) = 0$ and the estimation law evolves exponentially, towards a bounded value k^* as shown in fig. 6. At the same time the chaotic oscillator is forced onto the periodic orbit as shown in fig. 7. If we contrast these results with those in the previous example, we notice that the error still degrades to zero, even if no knowledge of the reference model nonlinearity has been supposed.

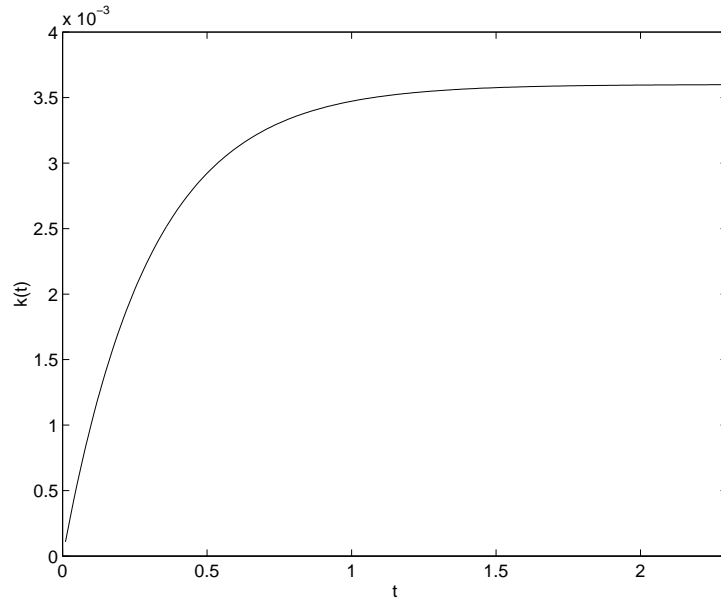


Figure 6: Evolution of the gain $k(t)$ under (14)

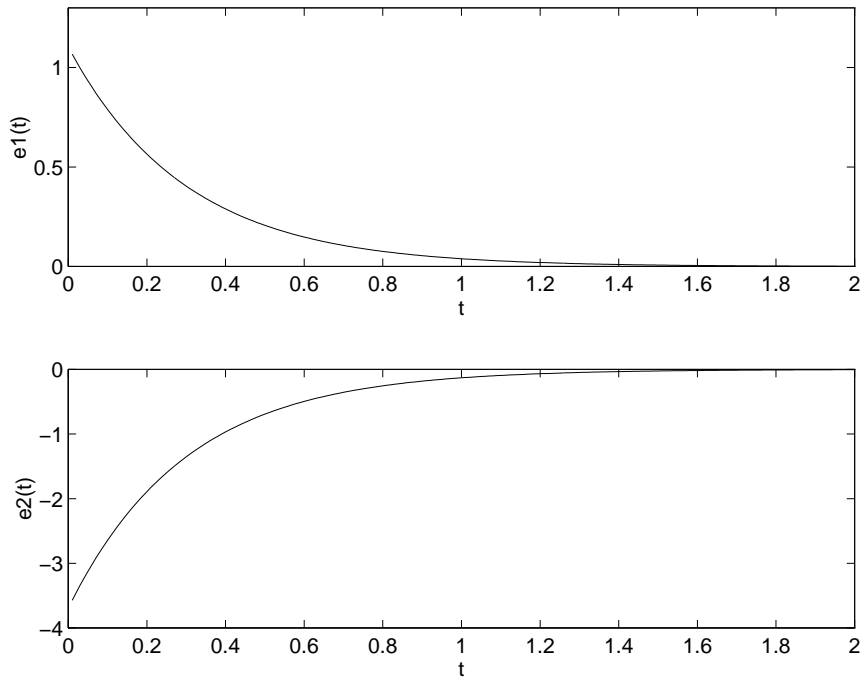


Figure 7: Error dynamic of the adaptive controlled Duffing Oscillator under (14)

5.3 An hypothesis on the nonlinearity of the system to control

All the controllers proposed up to now make use of a feedback linearization term to deal with the nonlinearity of the controlled system. To remove such an expensive action we now suppose that the projection h of f is bounded, up to an unknown scalar $\mu > 0$, by a known continuous function ϕ .

Assumption 3

$$\|h(x, t)\| \leq \mu\phi(x), \quad \forall (x, t) \in \mathcal{R}^n \times [0, \infty[; \quad (16)$$

Remark. Observe that this assumption is not as strong as it might seem. In many chaotic systems, for instance, the nonlinearities are polynomial, in which case (16) holds with $\phi(x) = \exp(\|x\|)$.

This assumption can be used to synthesize the controller :

$$\begin{cases} \dot{k}(t) = \|B^T P e\|(1 + \phi(x)) \\ u(t) = -K e - k(t)(1 + \phi(x))\|B^T P e\|^{-1} B^T P e \end{cases}, \quad (17)$$

where P is defined as in the previous section.

Here the feedback linearization term is not present anymore, and (16) has been used to define a new adaptation law able to bring the controlled system asymptotically onto the desired trajectory. In facts, if we now add the hypothesis of Theorem 3 to the statement of Assumption 3, we get the main result of this paper:

Theorem 4 *Let $P \in \mathcal{R}^{n \times n}$ be the positive definite solution of*

$$P\hat{L} + \hat{L}^T P + I = 0$$

and let

$$\dot{k}(t) = \|B^T P e(t)\|(1 + \phi(x(t))).$$

Define the mapping $D(e)$ as in Theorem 3 and embed the feedback controlled error system (4) in the differential inclusion :

$$\begin{cases} \dot{e} \in \{\hat{L}e + B[h(x(t), t) - l(y(x(t), t) + u) \mid u \in k(1 + \phi(e + y(t)))D(e)\} \\ \dot{k} \in \{\|B^T P e\|(1 + \phi(e + y))\} \end{cases}$$

Then, for every initial condition $(e(0), k(0)) = (e^0, k^0) \in \mathcal{R}^n \times \mathcal{R}$,

1. $\lim_{t \rightarrow \infty} k(t) = k^* < +\infty$;

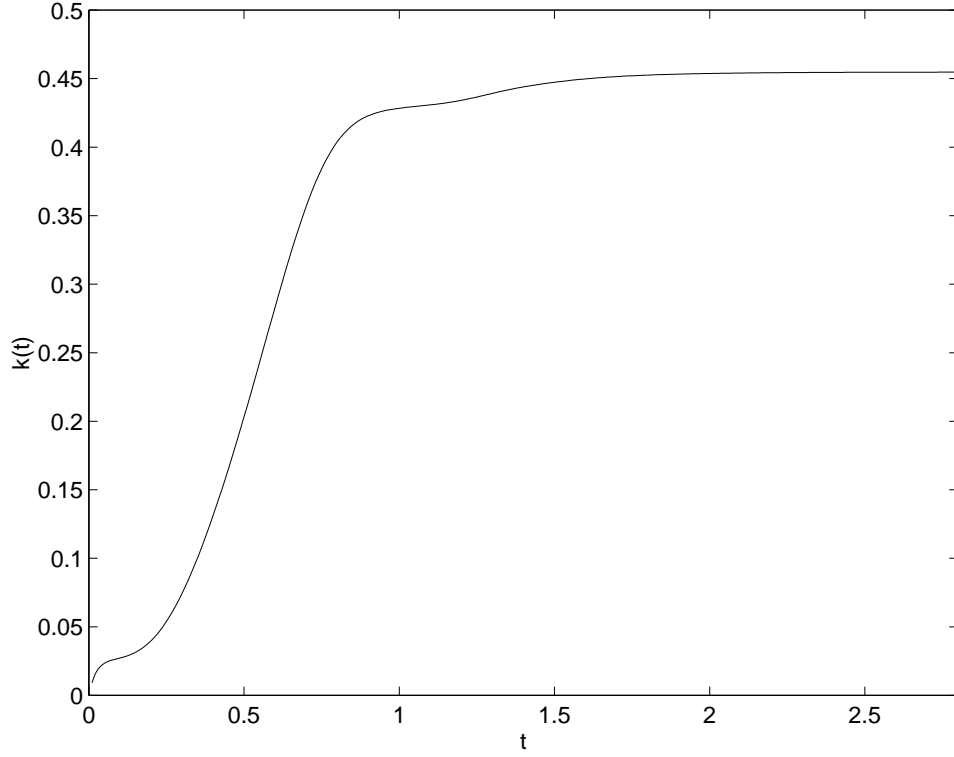


Figure 8: Evolution of gain $k(t)$

2. $\lim_{t \rightarrow \infty} e(t) = 0$.

Proof. The proof is analogous to that of Theorem 3 and therefore is omitted. \square

5.4 An adaptive controlled Duffing Oscillator

Given the Duffing Oscillator (6), we note that the nonlinearity involved is polynomial and therefore bounded by an exponential. Hence we choose

$$\phi(x) = e^{|x|}.$$

Then, we synthesize the adaptive control (17), having chosen the gain matrix K as in Sec. 4.3. In order to confirm that the result is global and show how the adaptation mechanism work, we run the simulation starting the systems from very distant initial conditions. Figs. 8,9 confirm the statement of Theorem4.

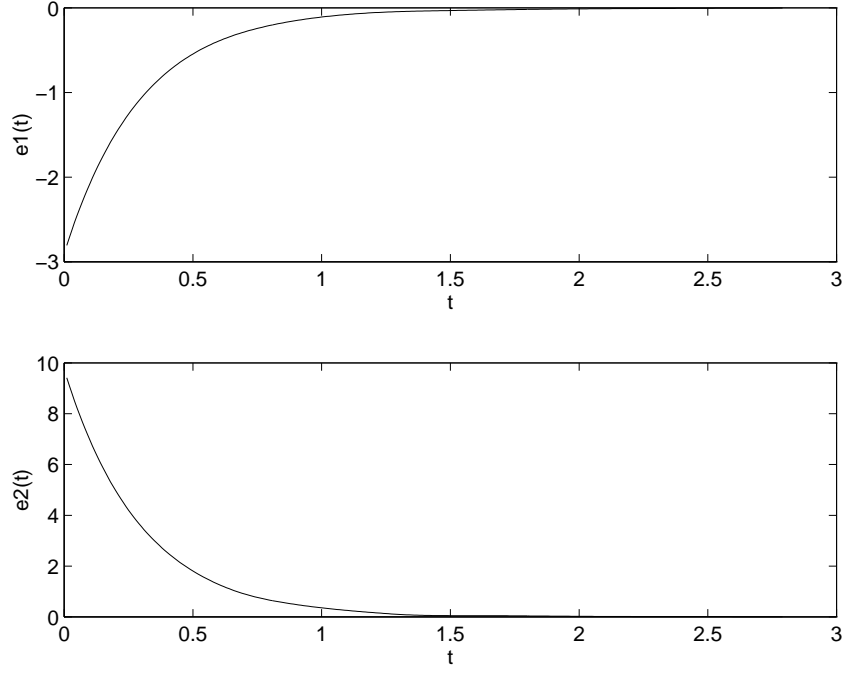


Figure 9: Error dynamic of the controlled Duffing Oscillator under (17)

6 An application : Synchronizing two Chua circuits

6.1 The Chua Circuit

The Chua circuit is a simple modified RLC circuit [Chua, Itoh, Kocarev & Eckert 1993]. It consists of a linear inductor L , a linear resistor R , two linear capacitors C_1 and C_2 and a nonlinear resistor N_R . The circuit equations can be written as

$$C_1 \frac{dv_{C_1}}{dt} = \frac{1}{R}(v_{C_2} - v_{C_1}) - g(v_{C_1}), \quad (18)$$

$$C_2 \frac{dv_{C_2}}{dt} = \frac{1}{R}(v_{C_1} - v_{C_2}) + i_L, \quad (19)$$

$$L \frac{di_L}{dt} = -v_{C_2}, \quad (20)$$

where $g(\cdot)$ is a *piecewise-linear* function defined by :

$$g(v_R) = G_b v_R + \frac{1}{2}(G_a - G_b)[|v_R + B_p| - |v_R - B_p|]$$

For convenience we rewrite the equations in a dimensionless form by rescaling the parameters of the system :

$$\begin{aligned} x &= \frac{v_{C_1}}{B_p} & y &= \frac{v_{C_2}}{B_p} & z &= \frac{i_L}{(B_p G)} & \tau &= \frac{t G}{C_2} \\ a &= R G_a & b &= R G_b & \alpha &= \frac{C_2}{C_1} & \beta &= \frac{C_2 R^2}{L} \end{aligned}$$

which gives the state equations

$$\dot{x} = \alpha(y - x - f(x)), \quad (21)$$

$$\dot{y} = x - y + z, \quad (22)$$

$$\dot{z} = -\beta y, \quad (23)$$

where

$$f(x) = bx + \frac{1}{2}(a - b)[|x + 1| - |x - 1|].$$

6.2 Synchronization

We first consider the synchronization of two identical chaotic systems [Chua et al. 1993]. In order to investigate such a problem we consider another Chua circuit, fixing the system parameters so that the circuits exhibit a chaotic attractor; specifically the so called *double - scroll* attractor. The following values produce the attractor shown in figs. 10,11,12 :

$$\begin{aligned} \alpha &= 10.00 & \beta &= 14.87 \\ a &= -1.27 & b &= -0.68 \end{aligned}$$

We consider next two Chua circuits starting from different initial conditions : $x'(0) = -2, y'(0) = 0.02, z'(0) = 4$ for the reference model and $x(0) = 0.7, y(0) = 0.4, z(0) = -0.8$ for the system we want to control. As we can see, because of the sensitive dependence on initial conditions, typical of any chaotic system, the two orbits diverge exponentially (figs. 10,11,12). In order to synchronize the two systems we follow the strategy outlined in Theorem 4. First we form the error equation

$$\dot{e} = \begin{pmatrix} -\alpha & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{pmatrix} e + \begin{pmatrix} \alpha[f(x') - f(x)] \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u$$

Notice that we are trying to achieve the synchronization by adding the control just to the first equation. Then we choose K as

$$K = (90 \quad 0.1 \quad 1)$$

so that $\hat{L} = L - BK$ has all eigenvalues in the left hand side of the complex plane. Now we have to find a function $\phi(x)$ that dominates the nonlinearity $f(x)$ of the Chua circuit we want to control. We notice that the nonlinearity $f(x)$ of Chua circuit is polynomial, hence :

$$\|f(x)\| \leq \phi(x) = e^{\|x\|}.$$

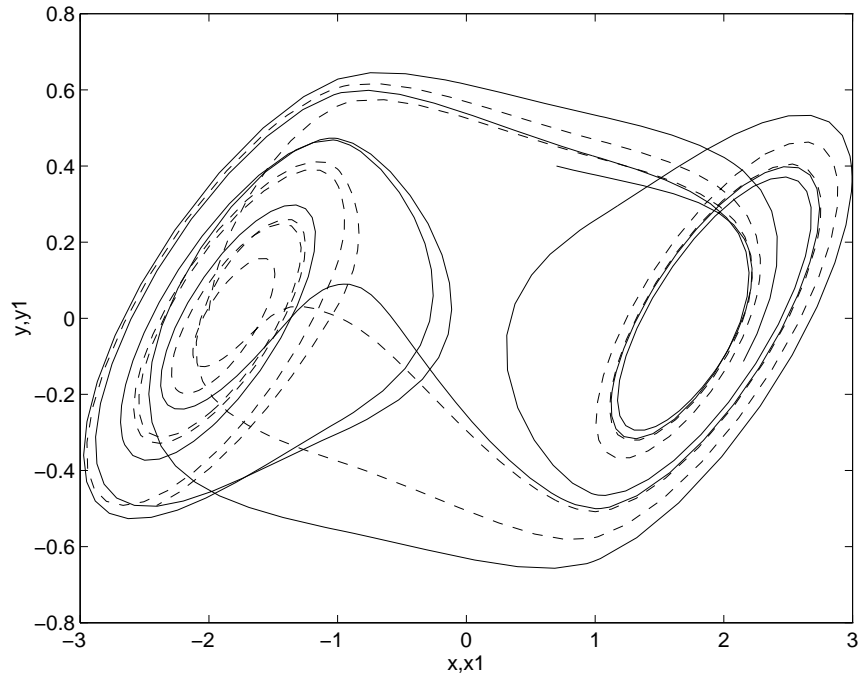


Figure 10: x-y projection of the Chua attractor (dashed line, reference; solid line, uncontrolled)

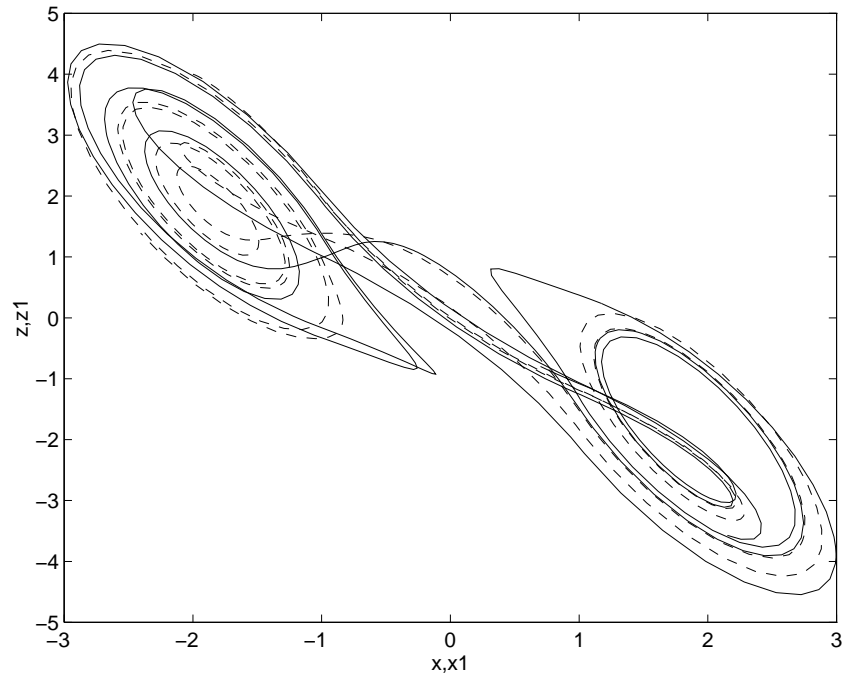


Figure 11: x-z projection of the Chua attractor (dashed line, reference; solid line, uncontrolled)

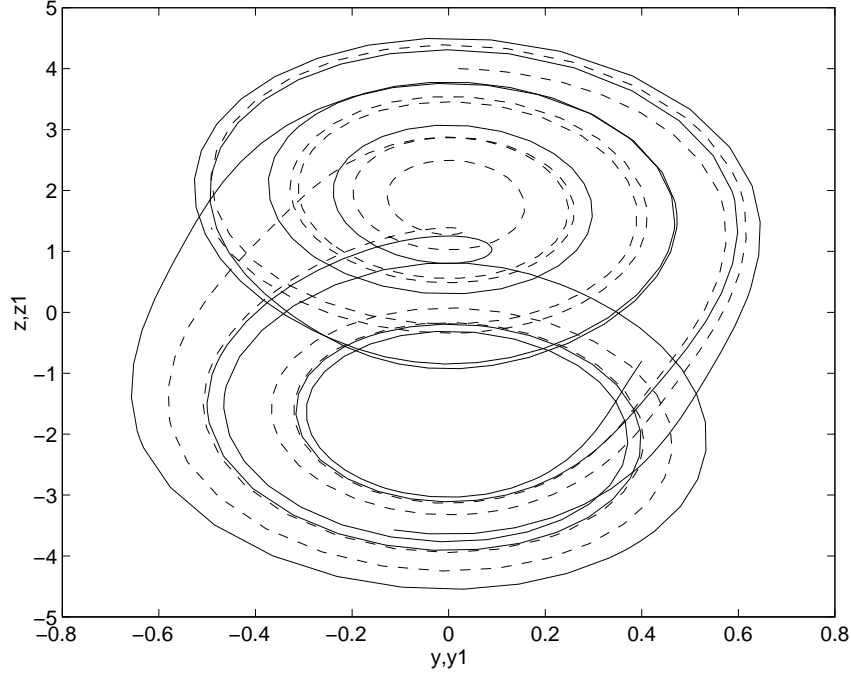


Figure 12: y-z projection of the Chua attractor (dashed line, reference; solid line, uncontrolled)

We now have everything to form the control (17) and we set

$$u(t) = -Ke - k(t)(1 + e^{\|x\|})\|B^T Pe\|B^T Pe \quad (24)$$

$$\dot{k} = \|B^T Pe\|(1 + e^{\|x\|}), \quad (25)$$

with P the solution of the Lyapunov equation (10).

By applying such a control we obtain the desired synchronization as shown in fig. 13. We notice that the dynamics of the synchronization errors are slower for the second and the third components of the state vectors. This is, indeed, a consequence of the fact that those states are controlled only indirectly, the control action being directed only to the first state.

Finally, fig. 14 shows that the gain $k(t)$ evolves towards a finite value as forecasted by Theorem 4.

7 Conclusion

Controlling chaos is a new, quickly evolving branch of control theory that will require far more research effort in order to be deeply understood and hopefully applied in real-world situations. In this paper, we have shown that it is possible to both control and synchronize a continuous-time

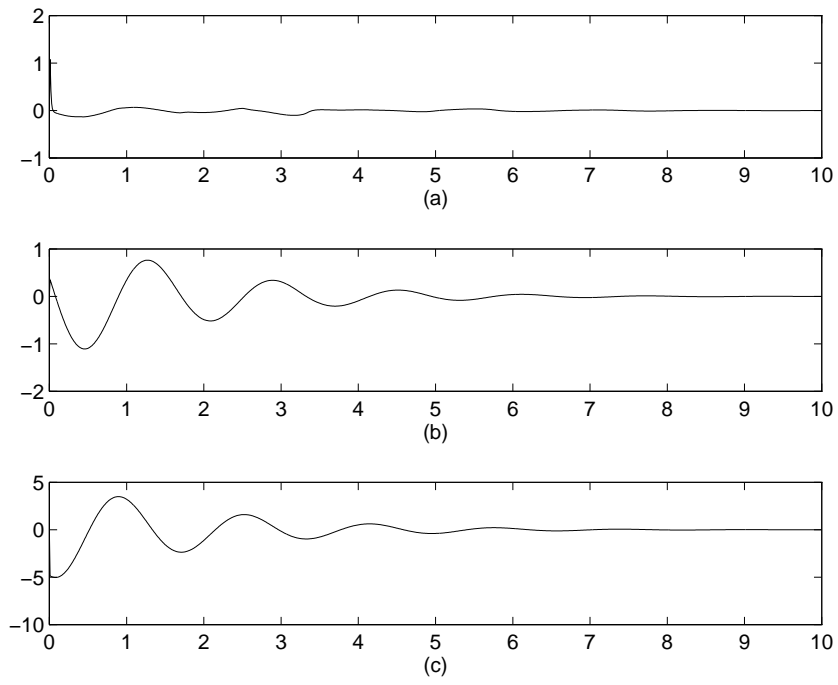


Figure 13: Error dynamics of the controlled Chua circuit

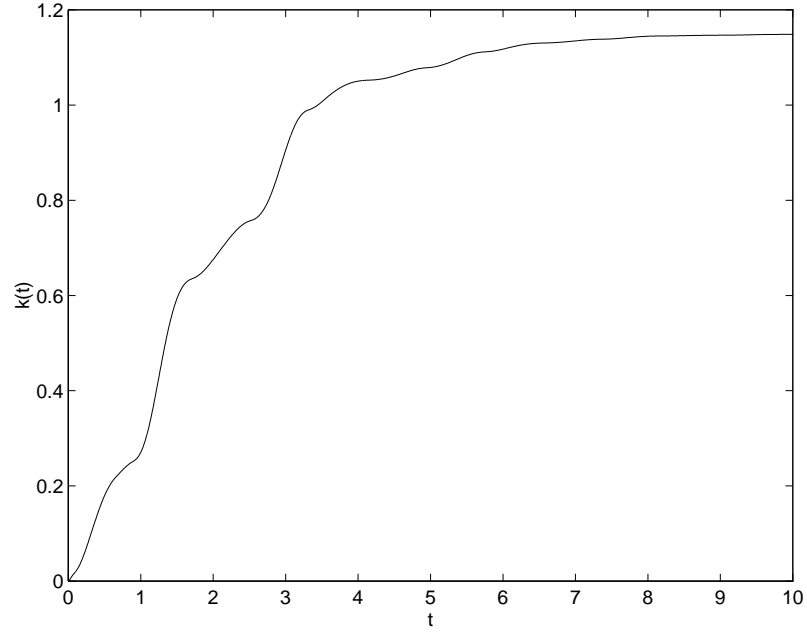


Figure 14: Evolution of the gain $k(t)$

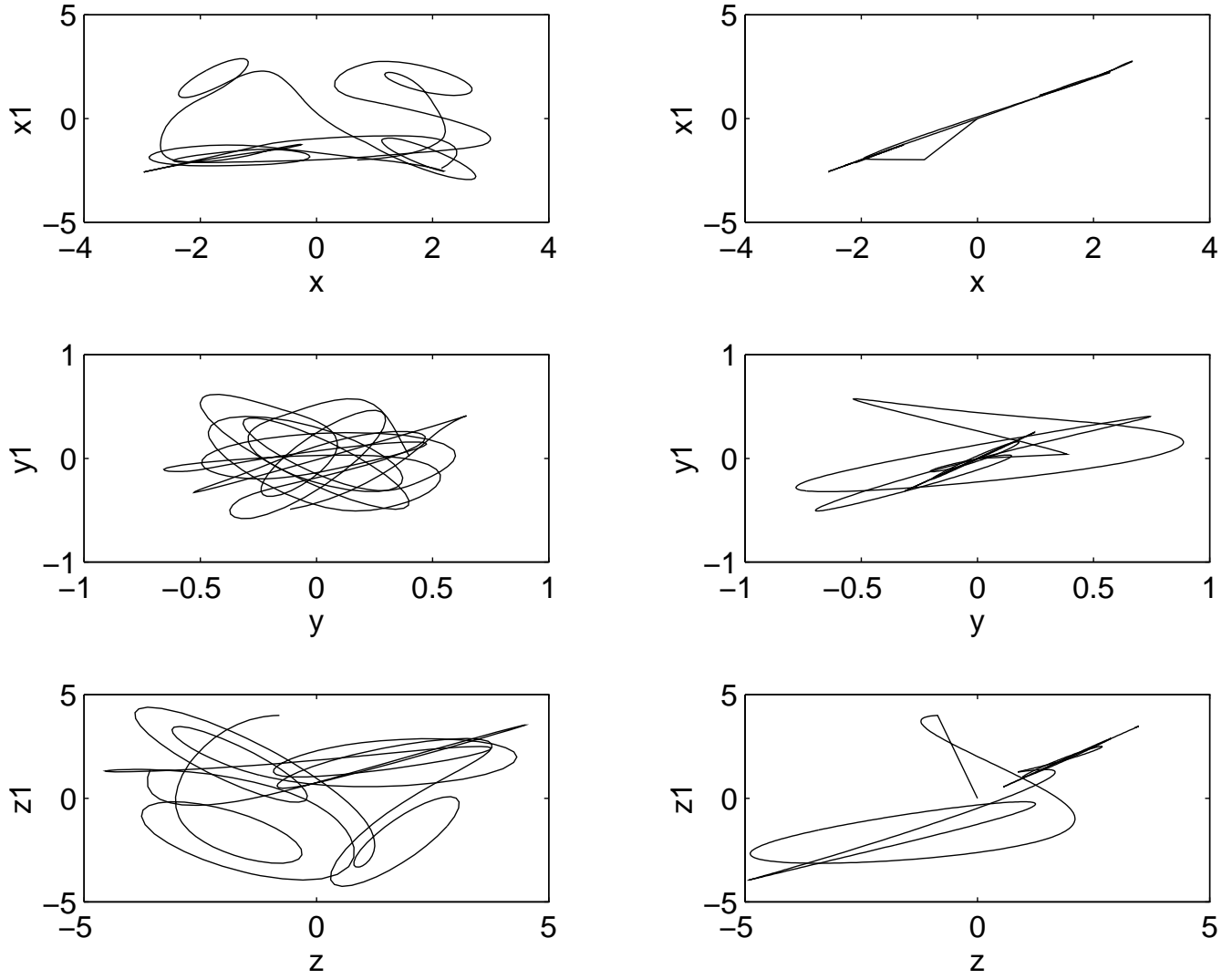


Figure 15: State relations before (left) and after (right) the control is applied

chaotic system by means of an adaptive strategy. In so doing, a special feature of chaos, namely the boundedness of chaotic attractors has been explicitly exploited, to synthesize a nonlinear reference-model-based control. Hence a classical engineering approach has been modified by using the fact that the systems involved are chaotic. Moreover the result obtained is global and it has been shown that the control can be achieved even by controlling only some of the system states. The synchronization of two identical Chua circuits, for instance, has been solved by adding a control term only to the first component of the state vector, without requiring any direct control of the other states.

However many things need still to be worked out. First of all, Assumption 1 should be removed and a controller should be found that does not require the error equation to have a linear component. A rigorous proof of a smooth adaptive controller, similar to the one proposed in Theorem 4 should be given. Finally, the robustness properties of this control strategy need to be investigated as well as its response in noisy environments. A first investigation of these problems seems to show that in presence of noise the synchronization is not lost, but the adaptively estimated gain is adjusted to a new bigger value. We conjecture that by limiting the upper value of the gain, the synchronization could still be obtained, up to a finite (small) synchronization error. A careful investigation of these effects is left for future work.

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